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BOUNDARY ESTIMATES OF p-HARMONIC FUNCTIONS IN A METRIC MEASURE SPACE

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1. Introduction

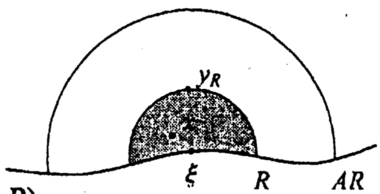
The purpose of this note is two-fold. First we discuss Carleson type estimates, which provide control of the bound of positive harmonic functions vanishing on a portion of the boundary. Such an estimate is well-known for harmonic functions in certain Euclidean domains. We shall prove a Carleson type estimate for p-harmonic functions on bounded John domains in a complete metric space equipped with an Ahlfors Q-regular measure supporting a $(1,p)$-Poincaré inequality for some $1 < p \le Q$. This part is based on [4].

Secondly, we discuss the Hölder continuity of p-harmonic functions up to the boundary. It is classical that a domain is regular, then the Dirichlet solution of a continuous boundary function is continuous up to the boundary. It may be natural to think that the better continuity of a boundary function ensures the better continuity of the Dirichlet solution. We shall investigate conditions on a domain for every Hölder continuous boundary function to have Hölder continuous solution with the same Hölder exponent. Our results are new even in the Euclidean setting when $p \neq 2$. This part is based on [5].

2. Carleson estimates for harmonic functions

Let us begin with the classical result due to Carleson.

Theorem A (Carleson [11]). *Let D be a bounded Lipschitz domain in R^n. Then there exists a constant $A > 1$ with the following property: for $\xi \in \partial D$ and $R > 0$ small, take a point $y_R \in D$ such that $|y_R - \xi| = R$ and $\operatorname{dist}(y_R, \partial D) \ge R/A$. Then*



$$u \le Au(y_R) \quad on\ D \cap B(\xi, R),$$

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whenever u is a positive harmonic function in $D \cap B(\xi, AR)$ with $u = 0$ on $\partial D \cap B(\xi, AR)$.

Ever since the Carleson's work there have been a large number of studies on this subjects. Most of them generalize the domain D and exploited harmonic analysis on non-smooth domains. There are several ways to prove the Carleson estimates in non-smooth domains:

(i) Carleson [11] and Jerison–Kenig [18] employed the uniform barrier. This argument requires the *Capacity Density Condition* for the complement of the domain.

(ii) In [1], the author prove the Carleson estimate by showing the *Boundary Harnack principle* first. The boundary Harnack principle was deduced from the estimates of the Green functions and representation of harmonic functions as the Green potential. This approach is not applicable to non-linear equations.

(iii) In the study of the Martin boundary of Denjoy domains, Benedicks [6] observed the Domar method [15] is useful. See Chevallier [13]. The Domar method is a very robust argument based on the sub-mean value property of subharmonic functions. In the sequel, we shall observe that the Domar method is applicable even to solutions of non-linear equations in metric measure spaces.

3. Metric measure space

Let (X, d, μ) be a proper metric measure space with doubling Borel measure μ. Here we say that X is proper if closed and bounded subsets of X are compact; and that μ is doubling if there is a constant $A \geq 1$ such that

$$\mu(B(x, 2r)) \leq A\mu(B(x, r)),$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ is the open ball with center x and radius r. For simplicity, we assume that X is Ahlfors Q-regular, i.e.,

$$A^{-1}r^Q \leq \mu(B(x, r)) \leq Ar^Q \quad \text{for every ball } B(x, r).$$

Throughout the note we fix $1 < p \leq Q$. We shall define the notion of p-harmonicity.

For a moment let f be a smooth function on R^n and let $\widetilde{xy}$ be a rectifiable curve. Then

$$|f(x) - f(y)| = \left| \int_{\widetilde{xy}} \nabla f \cdot dx \right| \leq \int_{\widetilde{xy}} |\nabla f| ds.$$

In view of this observation, Heinonen-Koskela [17] defined an *upper gradient* of a function f on a metric measure space X to be $g \geq 0$ such that for every rectifiable curve $\widetilde{xy} \subset X$

$$|f(x) - f(y)| \leq \int_{\widetilde{xy}} g ds. \tag{3.1}$$

The above requirement is somewhat too strong for the limiting operation. We say that g is a *weak upper gradient* of f if g satisfies (3.1) for all curves $\widetilde{xy}$ except for p-module zero. By g_f we denote the *minimal p-weak upper gradient* of f, i.e.,

$$g_f(x) := \inf_g \left(\limsup_{r \to 0^+} \fint_{B(x,r)} g d\mu \right).$$

The minimal p-weak upper gradient g_f satisfies (3.1) for all curves $\widetilde{xy}$ except for p-module zero. See [23] for these accounts. We assume the following $(1, p)$-Poincaré inequality.

Definition 1 (*$(1, p)$-Poincaré inequality*)**.** There exist constants $\kappa \geq 1$ (*scaling constant*) and $A_p \geq 1$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq A_p r \left(\fint_{B(x,\kappa r)} g_u^p d\mu \right)^{1/p}$$

whenever $B(x, r) \subset X$.

By the Hölder inequality $(1, q)$-Poincaré inequality with $q < p$ implies the $(1, p)$-Poincaré inequality. Conversely, Keith-Zhong [19] showed that if X supports a $(1, p)$-Poincaré inequality, then there is $q < p$ such that X supports a $(1, q)$-Poincaré inequality. Define the Sobolev space on X as follows.

Definition 2 (Sobolev or Newtonian space [23])**.** Define

$$\|u\|_{N^{1,p}} = \left(\int_X |u|^p d\mu \right)^{1/p} + \left(\int_X g_u^p d\mu \right)^{1/p}.$$

If $\|u - v\|_{N^{1,p}} = 0$, then we write $u \sim v$. The *Newtonian space* of X is the quotient

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}} < \infty\}/\sim$$

The space $N^{1,p}(X)$ equipped with the norm $\|\cdot\|_{N^{1,p}}$ is a Banach space and a lattice. Cheeger [12] gave an alternative definition of Sobolev space, which coincides with the above Newtonian space for $1 < p < \infty$. Moreover, the modulus of the Cheeger derivative and the minimum upper gradient are comparable:

$$A^{-1}|df(x)| \leq g_f(x) \leq A|df(x)|$$

([24, Corollary 3.7]). If $f = A$ on E, then $g_f = |df| = 0$ μ-a.e. on E ([12, Proposition 2.2]).

Definition 3. Define the *p-capacity* of $E \subset X$ by

$$\mathrm{Cap}_p(E) := \inf_u \left(\int_X |u|^p \, d\mu + \int_X |du|^p \, d\mu \right)$$

Here inf is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E. We say that a property holds p-q.e. if it holds except for E with $\mathrm{Cap}_p(E) = 0$.

Hereafter let $\Omega \subset X$ be a bounded domain in X with $\mathrm{Cap}_p(X \setminus \Omega) > 0$. The null-Sobolev space for Ω is defined by

$$N_0^{1,p}(\Omega) = \{u \in N^{1,p}(X) : u = 0 \ p\text{-q.e. on } X \setminus \Omega\}.$$

Definition 4. We say that u is *p-harmonic* in Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_U g_u^p \, d\mu \le \int_U g_{u+\varphi}^p \, d\mu$$

for all relatively compact subsets U of Ω and for every function $\varphi \in N_0^{1,p}(U)$. We say that u is *Cheeger p-harmonic* in Ω if $u \in N^{1,p}_{loc}(\Omega)$ and

$$\int_U |du|^p \, d\mu \le \int_U |d(u+\varphi)|^p \, d\mu$$

for all relatively compact subsets U of Ω and for every function $\varphi \in N_0^{1,p}(U)$. This is equivalent to the Euler equation:

$$\int_U |du|^{p-2} du \cdot d\varphi \, d\mu = 0.$$

Remark 1. If $p = 2$, then the above Euler equation is linear and hence Cheeger 2-harmonicity is a linear property. On the other hand, the p-harmonicity based on the upper gradient has no Euler equation and hence it is non-linear even if $p = 2$.

Definition 5. We say that u is a *p-subsolution* if

$$\int_U g_u^p \, d\mu \le \int_U g_{u+\varphi}^p \, d\mu$$

for all relatively compact subsets U of Ω and for every function $\varphi \in N_0^{1,p}(U)$. We say that u is a *p-quasiminimizer* if there exists $A_{qm} \ge 1$ such that

$$\int_U g_u^p \, d\mu \le A_{qm} \int_U g_{u+\varphi}^p \, d\mu$$

for all relatively compact subsets U of Ω and for every nonpositive function $\varphi \in N_0^{1,p}(U)$. If the inequality holds for every nonpositive function $\varphi \in N_0^{1,p}(U)$, then u is called *p-quasisubminimizer*.

It is easy to see that a Cheeger p-(sub)harmonic function is a p-quasi(sub)minimizer. Basic properties will be given for p-quasi(sub)minimizers, and hence p-(sub)harmonic functions and Cheeger p-(sub)harmonic functions can be treated simultaneously.

Definition 6. By $H_p^U f$ we denote the solution to the p-Dirichlet problem on the open set U with boundary data $f \in N^{1,p}(U)$, i.e., $H_p^U f$ is p-harmonic in U and $H_p^U f - f \in N_0^{1,p}(U)$. An upper semicontinuous function u is said to be p-subharmonic in Ω if the comparison principle holds, i.e., if $f \in N^{1,p}(U)$ is continuous up to ∂U and $u \le f$ on ∂U, then $u \le H_p^U f$ on U for all relatively compact subsets U of Ω.

Remark 2. We summarize functions:

(i) A (Cheeger) p-harmonic function is a p-quasiminimizer.
(ii) A (Cheeger) p-subsolution is a p-quasisubminimizer.
(iii) A bounded (Cheeger) p-subharmonic function is a p-quasisubminimizer.

4. Domar argument

Let $u \ge 0$ be a locally bounded p-quasisubminimizer. Then u is in the *De Giorgi class*, $DG_p(\Omega)$, i.e., if $B(x,R) \subset \Omega$, then

$$\int_{\{y \in B(x,\rho) : u(y) > k\}} g_u^p \, d\mu \le \frac{A}{(r-\rho)^p} \int_{\{y \in B(x,r) : u(y) > k\}} (u-k)^p \, d\mu$$

for every $k \in \mathbb{R}$ and $0 < \rho < r < R/\kappa$. Here g_u is the minimal p-weak upper gradient of u and $\kappa \ge 1$ is the scaling constant for the Poincaré inequality ([22, 20, 21]).

The above inequality is very strong; its repeated application, together with the De Giorgi method [14] yields the following estimate ([22]):

If $u \in DG_p(\Omega)$, $0 < R < \operatorname{diam}(X)/3$, $B(x,R) \subset \Omega$, then for every $k_0 \in \mathbb{R}$

$$\sup_{B(x,R/2)} u \le k_0 + A\Big(\fint_{B(x,R)} (u-k_0)_+^p d\mu \Big)^{1/p}.$$

Let $k_0 = 0$ and $u \ge 0$. We obtain the *weak submean value inequality*:

$$\text{(wsmv)} \qquad u(x) \le A_s \Big(\fint_{B(x,R)} u^p \, d\mu \Big)^{1/p}.$$

Here $A_s \ge 1$ is independent of x, R and u. This inequality may be regarded as a sort of the mean value inequality for p-subharmonic functions. Although it is weak ($A_s > 1$), it is sufficient to employ the Domar method and to give the Carleson estimate.

Lemma 1 ([15]). *Let Ω be a bounded open set and let $\delta_\Omega(x) = \operatorname{dist}(x, X\setminus\Omega)$. Suppose $u \geq 0$ locally bounded on Ω satisfies* (wsmv). *If there exists a positive constant ε such that*

$$I := \int_\Omega (\log^+ u)^{Q-1+\varepsilon}\, d\mu < \infty,$$

then

$$u(x) \leq A \exp(AI^{1/\varepsilon}\delta_\Omega(x)^{-Q/\varepsilon}) \quad \textit{for all } x \in \Omega.$$

Let us prepare the following estimate.

Lemma 2. *Suppose $u \geq 0$ satisfies* (wsmv) *and locally bounded on $B(x,R)$. Let $a > 2A_s$ and $0 < t \leq u(x)$. If*

$$\mu(\{y \in B(x,R) : \frac{t}{a} < u(y) \leq at\}) \leq \frac{\mu(B(x,R))}{a^{2p}},$$

then there exists a point $x' \in B(x,R)$ with $u(x') > at$.

Proof. Suppose $u \leq at$ on $B(x,R)$. Then (wsmv) gives

$$\begin{aligned} t &\leq \frac{A_s}{\mu(B(x,R))}\Big(\int_{B(x,R)\cap\{u\leq a^{-1}t\}} u(y)^p dy + \int_{B(x,R)\cap\{u>a^{-1}t\}} u(y)^p dy\Big)^{1/p} \\ &\leq A_s\left(\left(\frac{t}{a}\right)^p + \frac{(at)^p}{a^{2p}}\right)^{1/p} = \frac{2^{1/p}A_s}{a}t < 2^{1/p-1}t. \end{aligned}$$

This is a contradiction. □

Proof of Lemma 1. Observe $\mu(B(y,r)) \geq \dfrac{r^Q}{A_1}$ for $0 < r < 2\operatorname{diam}(\Omega)$. Let

$$R_j = (A_1 a^{2p}\mu(\{y \in \Omega : a^{j-2}u(x) < u(y) \leq a^j u(x)\}))^{1/Q}, \text{ which means}$$

$$\mu(\{y \in \Omega : a^{j-2}u(x) < u(y) \leq a^j u(x)\}) \leq \frac{R_j^Q}{A_1 a^{2p}} \leq \frac{\mu(B(x,R_j))}{a^{2p}}.$$

Then the lemma is proved by the following procedure:

- $\delta_\Omega(x) \leq 2\sum_{j=1}^\infty R_j$.
- $\sum_{j=1}^\infty R_j \leq AI^{1/Q}(\log^+ u(x))^{-\varepsilon/Q}$.
- $u(x) \leq \exp(AI^{1/\varepsilon}\delta_\Omega(x)^{-Q/\varepsilon})$.

Let us illustrate the most crucial step (i): Let $x_1 = x$, $t = u(x_1)$. If $\delta_\Omega(x_1) < R_1$, then STOP. Otherwise $B(x_1, R_1) \subset \Omega$, so

$$\begin{aligned}&\mu(\{y \in B(x_1, R_1) : a^{-1}u(x) < u(y) \le au(x)\}\\ &\le \mu(\{y \in \Omega : a^{-1}u(x) < u(y) \le au(x)\} \le \frac{\mu(B(x_1, R_1))}{a^{2p}}.\end{aligned}$$

By Lemma 2 we find $x_2 \in B(x_1, R_1)$ with $u(x_2) > au(x_1)$. If $\delta_\Omega(x_2) < R_2$, then STOP. Otherwise $B(x_2, R_2) \subset \Omega$, and we find $x_3 \in B(x_2, R_2)$ with $u(x_3) > au(x_2) > a^2u(x_1)$. Repeat the procedure. Since u is locally bounded above, $\{x_j\}$ is finite or $x_j \to \partial\Omega$. This gives $\delta_\Omega(x) \le 2\sum_{j=1}^{\infty} R_j$. □

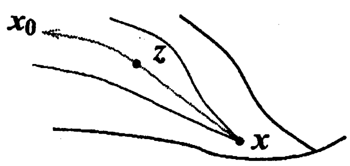
5. Carleson estimate for p-harmonic functions

A bounded domain D is called a *uniform domain* if for every couple of points $x, y \in D$ there exists a curve $\gamma \subset D$ connecting x and y such that

$$\begin{aligned}&\ell(\gamma) \le Ad(x, y),\\ &\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \le A\delta_\Omega(z) \quad (z \in \gamma).\end{aligned}$$

A Lipschitz domain and an NTA domain are uniform domains. Roughly speaking, a uniform domain is a domain satisfying the interior conditions for an NTA domain.

A bounded domain D is called a *John domain* with John center x_0 if the above condition holds with one fixed point $y = x_0$ and varying $x \in D$. Define the *quasi hyperbolic metric* by



$$k_D(x, y) = \inf_{\widetilde{xy}} \int_{\widetilde{xy}} \frac{ds}{\delta_D(z)},$$

where inf is taken over all curves $\widetilde{xy}$ connecting x and y in D. A John domain D satisfies the *quasihyperbolic boundary condition*

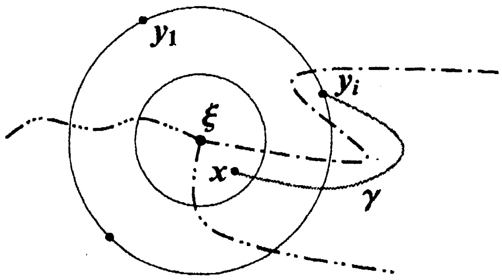
$$k_D(x, x_0) \le A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A.$$

This condition can be localized as follows.

Definition 7 (*Local reference points* [3]). A boundary point $\xi \in \partial D$ is said to have a *system of local reference points of order N* if there exist $R_\xi > 0$, $\lambda_\xi > 1$ and $A_\xi > 1$ with the following property: if $0 < R < R_\xi$, then we find

N points $y_1, \ldots, N \in D \cap S(\xi, R)$ such that $\delta_D(y_j) \geq R/A_\xi$ and such that for every $x \in D \cap \overline{B}(\xi, R/2)$ there is $i \in \{1, \ldots, N\}$ such that

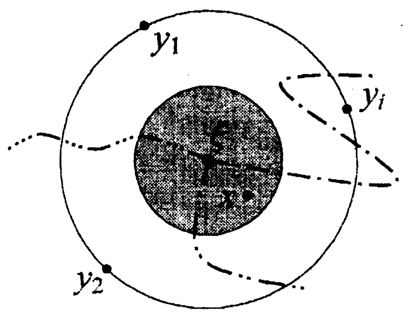
$$k_D(x, y_i) = k_{D \cap B(\xi, \lambda_\xi R)}(x, y_i) \leq A_\xi \left[\log\left(\frac{R}{\delta_D(x)} \right) + 1 \right].$$



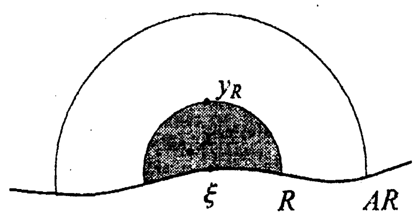
Remark 3. If D is a *uniform domain*, then every boundary point $\xi \in \partial D$ has a system of local reference points of *order 1*; the constants R_ξ, λ_ξ, A_ξ can be taken independently on ξ.

If D is a *John domain*, then there exists a finite number N such that each $\xi \in \partial D$ has a system of local reference points of *order N*; the constants R_ξ, λ_ξ, A_ξ can be taken independently on ξ. In general $N \geq 2$. If D is a *Denjoy domain*, then $N = 2$.

Theorem 1 (Carleson estimate for a John domain). *Let D be a John domain with $\xi \in \partial D$. For small $R > 0$ take local reference points $y_1, \ldots, y_N \in D \cap S(\xi, R)$. Suppose $h > 0$ is a bounded p-harmonic function on $D \cap B(\xi, 16R)$ with $h = 0$ on $\partial D \cap B(\xi, 16R)$. Then $h(x) \leq A \sum_{i=1}^{N} h(y_i)$ for $x \in D \cap B(\xi, R/4)$.*



Corollary 1 (Carleson estimate for a uniform domain). *Let D be a uniform domain with $\xi \in \partial D$. For small $R > 0$ take a nontangential point $y_R \in D \cap S(\xi, R)$, i.e., $\delta_D(y_R) \geq R/A$. Suppose $h > 0$ is a bounded p-harmonic function on $D \cap B(\xi, AR)$ with $h = 0$ on $\partial D \cap B(\xi, AR)$. Then $h(x) \leq Ah(y_R)$ for $x \in D \cap B(\xi, R)$. Here $A > 1$ depends only on D.*



Proof. Let us give a sketch of the proof. In view of the geometry of a uniform domain, we have

$$k_D(x, y_R) \leq A \log \frac{R}{\delta_D(x)} + A \quad \text{for } x \in D \cap B(\xi, AR).$$

Then the Harnack inequality gives

$$u(x) = \frac{h(x)}{h(y_R)} \le A\Big(\frac{R}{\delta_D(x)}\Big)^{\lambda}.$$

Extend u by $u = 0$ on $B(\xi, AR) \setminus D$. Then the extended function is a p-subsolution h on $\Omega = B(\xi, AR)$ with (wsmv).

An elementary geometrical observation gives

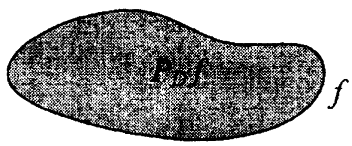
$$I = \int_\Omega \Big(\log^+\Big(\frac{h(x)}{h(y_R)}\Big)\Big)^{Q-1+\varepsilon} d\mu \le A \int_{D\cap B(\xi,AR)} \Big(\log^+\Big(\frac{R}{\delta_D(x)}\Big)^{\lambda}\Big)^{Q-1+\varepsilon} d\mu \le AR^Q.$$

Hence the Domar theorem yields

$$\frac{h(x)}{h(y_R)} = u(x) \le A\exp(AI^{1/\varepsilon}\delta_\Omega(x)^{-Q/\varepsilon}) \le A\exp(AR^{Q/\varepsilon}R^{-Q/\varepsilon}) = A$$

for $x \in D \cap B(\xi, R)$. See [4] for details. □

6. Hölder estimates of p-harmonic extension operators



Let $D \subset \mathbb{R}^n$ be a bounded open set and let f be a function on ∂D. Let $P_D f$ be the (Perron-Wiener-Brelot) Dirichlet solution of f over D. A boundary point $\xi \in \partial D$ is said to be regular if $\lim_{x\to\xi} P_D f(x) = f(\xi)$ for every $f \in C(\partial D)$. We say that D is a regular domain if every boundary point $\xi \in \partial D$ is regular. If D is regular, then P_D maps $C(\partial D)$ to $\mathcal{H}(D) \cap C(\overline{D})$. It is natural to raise the following question: Does the *better continuity* of a boundary function f guarantee the *better continuity* of $P_D f$?

An answer to this question was given in [2] for classical harmonic functions on Euclidean domains with Hölder continuity. In this note we investigate the same problem p-harmonic functions in metric measure space.

As was observed in the first part, the notions of p-harmonicity, p-Dirichlet problem, p-Perron solution, p-regularity, p-capacity, p-Wiener criterion are available (A. Björn, J. Björn, P. MacManus, and N. Shanmugalingam [10], [8], [9] and [7]).

Let $0 < \beta \le \alpha \le 1$. Consider the family $\Lambda_\alpha(E)$ of all bounded α-Hölder continuous functions u on E with norm

$$\|u\|_{\Lambda_\alpha(E)} := \sup_{x\in E}|u(x)| + \sup_{\substack{x,y\in E\\ x\ne y}} \frac{|u(x)-u(y)|}{d(x,y)^\alpha} < \infty.$$

We shall study the operator norm:

$$\|P_D\|_{\alpha\to\beta} := \sup_{\substack{f\in\Lambda_\alpha(\partial D)\\ \|f\|_{\Lambda_\alpha(\partial D)}\ne 0}} \frac{\|P_D f\|_{\Lambda_\beta(D)}}{\|f\|_{\Lambda_\alpha(\partial D)}}.$$

Heinonen, Kilpeläinen and Martio [16, Theorem 6.44] studied the condition for $\|P_D\|_{\alpha\to\beta} < \infty$ for $\beta < \alpha$ in Euclidean setting. The case most interesting case $\alpha = \beta$ has remained open.

7. Trivial boundary points

Is it true $\|P_D\|_{\alpha\to\beta} < \infty \implies D$ is p-regular?

This is not the case ([2]). A punctured ball D is p-irregular and yet $\|P_D\|_{\alpha\to\beta} < \infty$. To avoid such a pathological example we rule out *p-trivial boundary points*. We say that $a \in \partial D$ is a p-trivial boundary point if there is $r > 0$ such that $\mathrm{Cap}_p(\partial D \cap B(a,r)) = 0$.

Proposition 1. *Suppose $\|P_D\|_{\alpha\to\beta} < \infty$ for some $0 < \beta \le \alpha$. Then D is a p-regular domain if and only if ∂D has no p-trivial points.*

Hereafter let D be p-regular. Let $\alpha = \beta$. We shall study several conditions for $\|P_D\|_{\alpha\to\alpha} < \infty$. We have the *local or interior Hölder continuity* of p-harmonic functions ([22, Theorem 5.2]): There exists $\alpha_0 > 0$ such that every p-harmonic function in D is locally α_0-Hölder continuous in D. This constant α_0 depends only on p and the constants associated with the doubling property of μ and the Poincaré inequality, but not on D. In general, $\alpha_0 < 1$. In order to have $\|P_D\|_{\alpha\to\alpha} < \infty$, we *restrict ourselves to $\alpha \le \alpha_0$*.

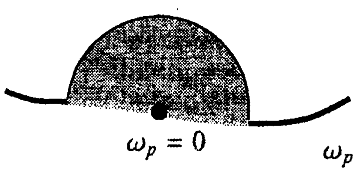
8. Relationships among several conditions

The conditions for $\|P_D\|_{\alpha\to\alpha} < \infty$ involve the *p-harmonic measure.*

Definition 8. By the *p-harmonic measure* $\omega_p(E;U)$ we mean the upper Perron solution $\overline{P}_U\chi_E$ of the boundary function χ_E in U ([9]).

Remark 4. The p-harmonic measure $\omega_p(E;U)$ need not be a measure unless $p = 2$ and the Cheeger harmonicity is adopted because of the non-linear nature of p-harmonicity.

Definition 9. *Global Harmonic Measure Decay Property: GHMD(α)*
We say that D satisfies the global harmonic measure decay property with exponent α if there exist $A_2 \ge 1$ and $r_0 > 0$ such that if $a \in \partial D$ and $0 < r < r_0$, then

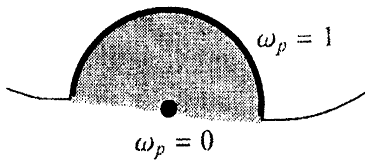


$$\omega_p(x;\partial D \setminus B(a,r), D) \le A_2 \left(\frac{d(x,a)}{r}\right)^{\alpha}$$

for all $x \in D \cap B(a,r)$.

Definition 10. *Local Harmonic Measure Decay Property: LHMD(α)*

We say that D satisfies the local harmonic measure decay property with exponent α if there exist $A_3 \geq 1$ and $r_0 > 0$ such that if $a \in \partial D$ and $0 < r < r_0$, then



$$\omega_p(x; D \cap S(a,r), D \cap B(a,r)) \leq A_3 \left(\frac{d(x,a)}{r}\right)^\alpha$$

for all $x \in D \cap B(a,r)$.

We shall use $\varphi_{a,\alpha}(x) = \min\{d(x,a)^\alpha, 1\}$ for $a \in \partial D$ as a test boundary function with respect to α-Hölder continuity.

Theorem 2. *Consider the following four conditions.*

(i) $\|P_D\|_{\alpha\to\alpha} < \infty$.
(ii) *There exists* A_4 *such that* $P_D\varphi_{a,\alpha}(x) \leq A_4 d(x,a)^\alpha$ *for all* $x \in D$.
(iii) *Global Harmonic Measure Decay of order* α.
(iv) *Local Harmonic Measure Decay of order* α.

Then we have

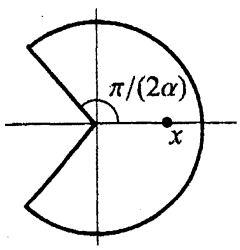
$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)} \iff \text{(iv)}.$$

If (iv) *holds for some* $\alpha' > \alpha$, *then* (i) *and* (ii) *hold.*

As an immediate corollary, we observe that the larger α is the stronger the property $\|P_D\|_{\alpha\to\alpha} < \infty$ is.

Corollary 2. *If* $0 < \beta \leq \alpha \leq \alpha_0$ *and* $\|P_D\|_{\alpha\to\alpha} < \infty$, *then* $\|P_D\|_{\beta\to\beta} < \infty$

Remark 5. It is not true that LHMD(α) $\implies$ $\|P_D\|_{\alpha\to\alpha} < \infty$. There is a domain D for which the LHMD(α) holds and yet $\|P_D\|_{\alpha\to\alpha} = \infty$. In fact let $D = \{z \in \mathbb{C} : |z| < 1, |\arg z| < \pi/(2\alpha)\}$ for $0 < \alpha \leq 1$. Then the LHMD(α) with respect to the classical harmonic measure holds. Nevertheless $\|P_D\|_{\alpha\to\alpha} = \infty$; if $\varphi(z) = |z|^\alpha$ for ∂D. then $\|\varphi\|_{\Lambda_\alpha(\partial D)} < \infty$ and yet $P_D\varphi(x) \approx x^\alpha \log(1/x)$ as $x \downarrow 0$ on the positive real axis, so $\|P_D\varphi\|_{\Lambda_\alpha(D)} = \infty$.



Let us consider some exterior conditions of the domain D in terms of the relative capacity:

$$\mathrm{Cap}_p(E, U) := \inf\Big\{\int_U g_u^p d\mu : u \in N_0^{1,p}(U) \text{ and } u \geq 1 \text{ on } E\Big\}.$$

Definition 11. We say that E is *uniformly p-fat* or satisfies the *p-capacity density condition* if there exist $A_5 > 0$ and $r_0 > 0$ such that

$$\frac{\mathrm{Cap}_p(E \cap B(a,r), B(a,2r))}{\mathrm{Cap}_p(B(a,r), B(a,2r))} \geq A_5$$

whenever $a \in E$ and $0 < r < r_0$.

Theorem 3. *The following five conditions are equivalent:*

(i) $\|P_D\|_{\alpha\to\alpha} < \infty$ *for some* $\alpha > 0$.
(ii) $P_D\varphi_{a,\alpha}(x) \le A_4 d(x,a)^\alpha$ *holds for some* $\alpha > 0$.
(iii) *GHMD(α) holds for some* $\alpha > 0$.
(iv) *LHMD(α) holds for some* $\alpha > 0$.
(v) $X \setminus D$ *satisfies the capacity density condition.*

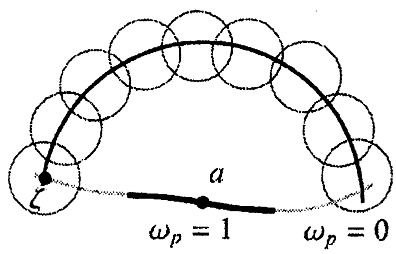
Corollary 3. *If* $X \setminus D$ *satisfies the* volume density condition:

$$\frac{\mu(B(a,r)\setminus D)}{\mu(B(a,r))} \ge A, \quad \text{for every } a \in \partial D \text{ and } < r < r_0,$$

then $\|P_D\|_{\alpha\to\alpha} < \infty$ *for some* $\alpha > 0$.

Remark 6. Our arguments are based mostly on the *comparison principle* for p-harmonic functions and the variational properties of the *De Giorgi class*, which includes p-harmonic functions. The crucial part is *GHMD* $\Longrightarrow$ *LHMD* for which we need the refinement of *the submean value property* for the De Giorgi class.

Remark 7. The comparison principle implies LHMD $\Longrightarrow$ GHMD. The converse implication GHMD $\Longrightarrow$ LHMD is crucial. Let us illustrate its proof:



Let $u = \omega_p(\partial D \cap B(a,r); D)$. Suppose $\zeta \in \partial D \cap S(a, Ar)$ (*UP*). Then $u \le \frac{1}{2}$ on $B(\zeta, cr)$, so $u \le 1 - \varepsilon$ on a small ball intersecting $B(\zeta, cr)$ by some argument based on the De Giorgi class. Repeating the same argument, we obtain $u \le 1 - \varepsilon$ on $S(a, Ar)$. Hence $\omega_p(\partial D \setminus B(a,r); D) \ge \varepsilon$ on $D \cap S(a, Ar)$; in other words

$$\omega_p(D \cap S(a,Ar); D \cap B(a,Ar)) \le \varepsilon^{-1}\omega_p(\partial D \setminus B(a,r); D) \text{ on } D \cap B(a,Ar).$$

Hence *GHMD* $\Longrightarrow$ *LHMD.*

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